

PROBABILISTIC AND AVERAGE LINEAR WIDTHS OF WEIGHTED SOBOLEV SPACES ON THE BALL EQUIPPED WITH A GAUSSIAN MEASURE

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ABSTRACT. Let $L_{q,\mu}$, $1 \leq q \leq \infty$, denotes the weighted L_q space of functions on the unit ball \mathbb{B}^d with respect to weight $(1 - \|x\|_2^2)^{\mu - \frac{1}{2}}$, $\mu \geq 0$, and let $W_{2,\mu}^r$ be the weighted Sobolev space on \mathbb{B}^d with a Gaussian measure ν . We investigate the probabilistic linear (n, δ) -widths $\lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu})$ and the p -average linear n -widths $\lambda_n^{(a)}(W_{2,\mu}^r, \mu, L_{q,\mu})_p$, and obtain their asymptotic orders for all $1 \leq q \leq \infty$ and $0 < p < \infty$.

1. INTRODUCTION

Let K be a bounded subset of a normed linear space X with norm $\|\cdot\|_X$. The linear and Kolmogorov n -widths of the set K in X are defined by

$$\lambda_n(K, X) := \inf_{T_n} \sup_{x \in K} \|x - L_n x\|_X,$$

and

$$d_n(K, X) := \inf_{F_n} \sup_{x \in K} \inf_{y \in F_n} \|x - y\|_X,$$

respectively, where T_n runs over all linear operators from X to X with rank at most n , and F_n runs through all possible linear subspaces of X with dimension at most n . They reflect the optimal errors of the “worst” elements of K in the approximation by linear operators with rank n and n -dimensional subspaces.

Now let W be a separable Banach space and assume that W contains a Borel field \mathcal{B} consisting of open subsets of W and is equipped with a probability measure γ defined on \mathcal{B} . For $0 < p < \infty$, the p -average linear and Kolmogorov n -widths are defined by

$$\lambda_n^{(a)}(W, \gamma, X)_p = \inf_{L_n} \left(\int_W \|x - L_n x\|_X^p \gamma(dx) \right)^{1/p},$$

and

$$d_n^{(a)}(W, \gamma, X)_p = \inf_{F_n} \left(\int_W \inf_{y \in F_n} \|x - y\|_X^p \gamma(dx) \right)^{1/p},$$

respectively. They reflect the optimal approximation of “most” elements of classes by linear operators with rank n and n -dimensional subspaces. We stress that for a centered Gaussian measure, the averaging parameter p is irrelevant up to a constant (see [12, Theorem 1.2] or [29, Corollary 1]).

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Let $\delta \in [0, 1]$. The probabilistic linear and Kolmogorov (n, δ) -widths of a set W with a measure γ in the space X are defined by

$$\lambda_{n,\delta}(W, \gamma, X) = \inf_{G_\delta} \lambda_n(W \setminus G_\delta, X),$$

and

$$d_{n,\delta}(W, \gamma, X) = \inf_{G_\delta} d_n(W \setminus G_\delta, X),$$

respectively, where G_δ runs through all possible measurable subsets in W with measure $\gamma(G_\delta) \leq \delta$. Hence, the probabilistic linear and Kolmogorov (n, δ) -widths can be understood as the γ -distribution of the approximation on all measurable subsets of W by linear operators with rank n and by n -dimensional subspaces. Therefore, the probabilistic case setting reflects the intrinsic structure of the class, and compared with the worst case setting, allows one to give deeper analysis of the approximation for the function class.

This paper is devoted to discussing the average and probabilistic linear widths of weighted Sobolev spaces on the unit ball with a Gaussian measure. Let $\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$ denote the unit ball in \mathbb{R}^d , where $x \cdot y$ is the usual inner product, and $\|x\|_2 = (x \cdot x)^{1/2}$ is the usual Euclidean norm. For the weight $W_\mu(x) = (1 - \|x\|_2^2)^{\mu-1/2}$ ($\mu \geq 0$), denote by $L_{p,\mu} \equiv L_p(\mathbb{B}^d, W_\mu(x) dx)$, $1 \leq p < \infty$, the space of measurable functions defined on \mathbb{B}^d with the finite norm

$$\|f\|_{p,\mu} := \left(\int_{\mathbb{B}^d} |f(x)|^p W_\mu(x) dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

and for $p = \infty$ we assume that $L_{\infty,\mu}$ is replaced by the space $C(\mathbb{B}^d)$ of continuous functions on \mathbb{B}^d with the uniform norm.

We denote by Π_n^d the space of all polynomials in d variables of degree at most n , and by \mathcal{V}_n^d the space of all polynomials of degree n which are orthogonal to polynomials of low degree in $L_{2,\mu}$. Note that

$$a_n^d := \dim \mathcal{V}_n^d = \binom{n+d-1}{n} \asymp n^{d-1}.$$

It is well known (see [9, p. 38 or p. 229]) that the spaces \mathcal{V}_n^d are just the eigenspaces corresponding to the eigenvalues $-n(n+2\mu+d-1) =: -\lambda_n$ of the second-order differential operator

$$D_\mu := \Delta - (x \cdot \nabla)^2 - (2\mu + d - 1) x \cdot \nabla,$$

where the Δ and ∇ are the Laplace operator and gradient operator respectively. More precisely,

$$D_\mu P = -n(n+2\mu+d-1)P = -\lambda_n P \quad \text{for } P \in \mathcal{V}_n^d.$$

Also, the spaces \mathcal{V}_n^d are mutually orthogonal in $L_{2,\mu}$ and

$$(1.1) \quad L_{2,\mu} = \bigoplus_{n=0}^{\infty} \mathcal{V}_n^d, \quad \Pi_n^d = \bigoplus_{k=0}^n \mathcal{V}_k^d.$$

Let

$$\{\phi_{nk} \equiv \phi_{nk}^d \mid k = 1, \dots, a_n^d\}$$

be a fixed orthonormal basis for \mathcal{V}_n^d . Then we know that

$$\{\phi_{nk} \mid k = 1, \dots, a_n^d, n = 0, 1, 2, \dots\}$$

is an orthonormal basis for $L_{2,\mu}$ with inner product

$$\langle f, g \rangle := \int_{\mathbb{B}^d} f(x)g(x)W_\mu(x) dx.$$

Denote by S_n the orthogonal projector of $L_{2,\mu}$ onto Π_n^d , which is called the Fourier partial summation operator. Evidently, for any $f \in L_{2,\mu}$, (1.1) can be rewritten in the form

$$f = \sum_{n=0}^{\infty} Proj_n f, \quad S_n(f) := \sum_{k=0}^n Proj_k f,$$

where $Proj_n$ is the orthogonal projector from $L_{2,\mu}$ onto \mathcal{V}_n^d and can be written as

$$(1.2) \quad Proj_n(f)(x) = \sum_{k=1}^{a_n^d} \langle \phi_{nk}, f \rangle \phi_{nk}(x) = \int_{\mathbb{B}^d} f(y) P_n(x, y) W_\mu(y) dy,$$

where $P_n(x, y) = \sum_{k=1}^{a_n^d} \phi_{nk}(x) \phi_{nk}(y)$ is the reproducing kernel for \mathcal{V}_n^d . It is known that for $\mu > 0$, the kernel $P_n(x, y)$ has the compact representation (see [32])

$$(1.3) \quad P_n(x, y) = b_d^\mu b_1^{\mu-\frac{1}{2}} \frac{n+\lambda}{\lambda} \int_{-1}^1 C_n^\lambda((x, y) + u \sqrt{1-\|x\|_2^2} \sqrt{1-\|y\|_2^2}) (1-u^2)^{\mu-1} du.$$

Here, C_n^λ is the n -th degree Gegenbauer polynomial, $\lambda = \mu + \frac{d-1}{2}$, $b_d^\gamma := (\int_{\mathbb{B}^d} (1-\|x\|_2^2)^{\gamma-1/2} dx)^{-1}$. See [32] for the proof of the formula of $P_n(x, y)$, including the limiting case of $\mu = 0$.

Given $r \in \mathbb{R}$, we define the fractional power $(-D_\mu^d)^{r/2}$ of the operator $-D_\mu^d$ on f by

$$(-D_\mu^d)^{r/2}(f) = \sum_{k=1}^{\infty} (k(k+2\mu+d-1))^{r/2} Proj_k(f) = \sum_{k=1}^{\infty} \lambda_k^{r/2} Proj_k(f),$$

in the sense of distribution. We call $f^{(r)} := (-D_\mu^d)^{r/2}(f)$ the r -th-order derivative of the distribution f .

For $r > 0$, the weighted Sobolev space $W_{2,\mu}^r \equiv W_{2,\mu}^r(\mathbb{B}^d)$ is defined by

$$W_{2,\mu}^r := \left\{ f = \sum_{n=1}^{\infty} Proj_n(f) = \sum_{n=1}^{\infty} \sum_{k=1}^{a_n^d} \langle \phi_{nk}, f \rangle \phi_{nk} \mid \int_{\mathbb{B}^d} f(x) W_\mu(x) dx = 0, \right. \\ \left. \|f\|_{W_{2,\mu}^r}^2 := \langle f^{(r)}, f^{(r)} \rangle = \sum_{n=1}^{\infty} \lambda_n^r \|Proj_n f\|_{2,\mu}^2 = \sum_{n=1}^{\infty} \lambda_n^r \sum_{k=1}^{a_n^d} |\hat{f}_{nk}|^2 < \infty \right\}$$

with inner product

$$\langle f, g \rangle_r := \langle f^{(r)}, g^{(r)} \rangle.$$

Obviously, it is a Hilbert space. If $1 \leq q \leq \infty$, $r > (d+2\mu)(\frac{1}{2} - \frac{1}{q})_+$, then the space $W_{2,\mu}^r$ can be continuously embedded into the space $L_{q,\mu}$ (see [29, Lemma 1]).

We equip $W_{2,\mu}^r$ with a Gaussian measure ν whose mean is zero and whose correlation operator C_ν has eigenfunctions ϕ_{lk} , $k = 1, \dots, a_l^d$, $l = 1, 2, \dots$ and eigenvalues

$$\nu_l = \lambda_l^{-s/2}, \quad s > d,$$

that is,

$$C_\nu \phi_{lk} = \lambda_l^{-s/2} \phi_{lk}, \quad k = 1, \dots, a_l^d, \quad l = 1, 2, \dots$$

Then (see [2, pp. 48-49]),

$$\langle C_\nu f, g \rangle_r = \int_{W_{2,\mu}^r} \langle f, h \rangle_r \langle g, h \rangle_r \nu(dh).$$

Denote by $(W_{2,\mu}^r)^*$ the space of all continuous linear functionals on $W_{2,\mu}^r$, and by $L_2(W_{2,\mu}^r, \nu)$ the usual space of ν -measurable functions ϕ on $W_{2,\mu}^r$ with finite norm

$$\|\phi\|_{L_2(\nu)} := \left(\int_{W_{2,\mu}^r} |\phi(x)|^2 \nu(dx) \right)^{1/2}.$$

Then $(W_{2,\mu}^r)^* = W_{2,\mu}^r$ can be embedded into $L_2(W_{2,\mu}^r, \nu)$. Put (see [2, p. 44])

$$H(\nu) = \left\{ g \in W_{2,\mu}^r : |g|_{H(\nu)} := \sup_{f \in (W_{2,\mu}^r)^* = W_{2,\mu}^r, R_\nu(f)(f) \leq 1} |\langle f, g \rangle| < \infty \right\},$$

where

$$R_\nu(f)(g) := \int_{W_{2,\mu}^r} \langle h, f \rangle_r \langle h, g \rangle_r \nu(dh), \quad f, g \in (W_{2,\mu}^r)^* = W_{2,\mu}^r.$$

The space $H(\nu)$ is called the **Cameron-Martin space** (or the reproducing kernel Hilbert space) of ν . Set $\rho = r + s/2$. It is easy to see from [2, pp. 48-49] that the Cameron-Martin space $H(\nu)$ of the Gaussian measure ν is $W_{2,\mu}^\rho$, i.e.,

$$(1.4) \quad H(\nu) = W_{2,\mu}^\rho \quad \text{and} \quad (\cdot, \cdot)_{H(\nu)} = \langle \cdot, \cdot \rangle_\rho.$$

Then the covariance of ν

$$(1.5) \quad R_\nu(f)(g) = \int_{W_{2,\mu}^r} \langle f, h \rangle_r \langle g, h \rangle_r \nu(dh) = \langle C_\nu f, g \rangle_r = \langle C_\nu f, C_\nu g \rangle_\rho.$$

For any fixed $f_1, \dots, f_n \in W_{2,\mu}^r$, the random vector $(\langle f, f_1 \rangle_r, \dots, \langle f, f_n \rangle_r)$ on the measurable space $(W_{2,\mu}^r, \nu)$ has the centered Gaussian distribution with covariance matrix $(\langle C_\nu f_i, f_j \rangle_r)_{i,j=1,\dots,n} = (\langle C_\nu f_i, C_\nu f_j \rangle_\rho)_{i,j=1,\dots,n}$. In special, on the cylindrical subsets, the measure ν has the form: let $g_k, k = 1, 2, \dots, n$ be any orthonormal system of functions in $L_{2,\mu}$, $g_k \in H(\nu) = W_{2,\mu}^\rho$, and let D be an arbitrary Borel subset of \mathbb{R}^n , then the measure of the cylindrical subset

$$G = \left\{ f \in W_{2,\mu}^r \mid (\langle f, g_1^{(\rho)} \rangle, \dots, \langle f, g_n^{(\rho)} \rangle) \in D \right\}$$

is equal to

$$\mu(G) = (2\pi)^{-\frac{n}{2}} \int_D \exp\left(-\frac{1}{2} \sum_{l=1}^n u_l^2\right) du_1 \cdots du_n.$$

See [2] and [13] for more information about Gaussian measure on Banach spaces. Throughout the paper, we always suppose that ν is the above Gaussian measure on $W_{2,\mu}^r$, $r > (d + 2\mu)(\frac{1}{2} - \frac{1}{q})_+$, $s > d$, and $\rho = r + s/2$.

The aim of the paper is to study the average and probabilistic linear widths of $W_{2,\mu}^r$ with the measure ν in $L_{q,\mu}$. Probabilistic and average widths has been studied only recently and are closely related with some other different problems, such as ε -complexity and the minimal error of problems of function approximation and integration by using standard or general linear information, in the probabilistic and average case setting (see [24], [22]). A few interesting results have been obtained. These include results on probabilistic and average Kolmogorov and linear widths of a Sobolev space of single-variate periodic functions with a Gaussian measure and

the $C^r[0, 1]$ space equipped with the r -fold Wiener measure in the L_q norm for $1 \leq q \leq \infty$ (see [10, 11, 14, 15, 16, 17, 18, 19, 23, 25]), of a space of multivariate periodic Sobolev functions equipped with a Gaussian measure in the L_q norm for $1 < q < \infty$ (see [3, 4, 28]), and of a Sobolev space $W_2^r(\mathbb{M}^{d-1})$ with a Gaussian measure on compact two-point homogeneous spaces \mathbb{M}^{d-1} for $1 \leq q \leq \infty$ (see [26]). In the worst and average case setting, the orders of the Kolmogorov and linear widths of weighted Sobolev classes on \mathbb{B}^d in $L_{q,\mu}$ were presented in [27] and [29] respectively. More information about average and probabilistic case setting results can be found in [22] and [24].

In this paper, we investigate probabilistic and average linear widths of the weighted Sobolev space $W_{2,\mu}^r$ with the measure ν in $L_{q,\mu}$, and obtain the sharp estimation. Our main results (Theorems 1.1 and 1.2 below) can be formulated as follows:

Theorem 1.1. *Let $\delta \in (0, 1/2]$, $1 \leq q \leq \infty$, and let $\rho > d/2 + 2\mu d(1/2 - 1/q)_+$. Then*

$$(1.6) \quad \lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu}) \asymp \begin{cases} n^{-\frac{\rho}{d} + \frac{1}{2}} (1 + n^{-\min\{\frac{1}{2}, \frac{1}{q}\}} (\ln(\frac{1}{\delta}))^{\frac{1}{2}}), & 1 \leq q < \infty, \\ n^{-\frac{\rho}{d} + \frac{1}{2}} (\ln(n/\delta))^{1/2}, & q = \infty, \end{cases}$$

where $A(n, \delta) \asymp B(n, \delta)$ means $A(n, \delta) \ll B(n, \delta)$ and $B(n, \delta) \ll A(n, \delta)$, $A(n, \delta) \ll B(n, \delta)$ means that there exists a positive constant c independent of n and δ such that $A(n, \delta) \leq cB(n, \delta)$ for any $n \in \mathbb{N}$ and $\delta \in (0, 1/2]$.

Theorem 1.2. *Let $0 < p < \infty$, $1 \leq q \leq \infty$, and let ρ be given as in Theorem 1.1. Then*

$$(1.7) \quad \lambda_n^{(a)}(W_{2,\mu}^r, \nu, L_{q,\mu})_p \asymp \begin{cases} n^{-\rho/d + 1/2}, & 1 \leq q < \infty, \\ n^{-\rho/d + 1/2} \sqrt{\ln(en)}, & q = \infty. \end{cases}$$

Theorem 1.2 extends (2.17) in [29] which gave the orders of $\lambda_n^{(a)}(W_{2,\mu}^r, \nu, L_{q,\mu})_p$ for $1 \leq q < 2 + 1/\mu$ and $0 < p < \infty$. The proof of Theorem 1.2 is different from the one in [29]. We use Theorem 1.1 to prove Theorem 1.2. In order to prove Theorem 1.1, we also use the discretization method as in the previous works concerning probabilistic widths. However, unlike the periodic case and the Wiener measure case, we cannot find an orthogonal transform between a finite-dimensional function space and a sequence space of function values with comparable L_q and ℓ_q norm, so we cannot use the invariant properties of the standard Gaussian measure under action of an orthogonal transform to give the discretization theorems. Instead, we use the comparison theorems for Gaussian measures. Unlike the spherical case, we cannot find a equivalent-weight Marcinkiewicz-Zygmund (MZ) inequalities on the ball (see Remark 3 in [27]), so we cannot use known results on the probabilistic linear widths of the identity matrix on \mathbb{R}^m . Instead, we need to find out and use the probabilistic linear widths of diagonal matrixes on \mathbb{R}^m .

Remark 1.3. Let $BH(\nu)$ be the unit ball of the Cameron-Martin space of the Gaussian measure ν . Then $BH(\nu) = BW_{2,\mu}^\rho$. It follows from [27] that the classical Kolmogorov and linear widths of $BH(\nu)$ in $L_{q,\mu}$ have the same error order for $1 \leq q \leq 2$, however, for $q > 2$, the Kolmogorov width $d_n(BH(\nu), L_{q,\mu})$ is essentially less than the linear width $d_n(BH(\nu), L_{q,\mu})$, and optimal linear operators lose to optimal nonlinear operators by a factor $cn^{1/2-1/q}$ in the worst case setting. From [29, (2.16)] and (1.7), we know that in the average case setting, the Kolmogorov and

linear widths of $W_{2,\mu}^r$ in $L_{q,\mu}$ have the same error order for $1 \leq q < \infty$. This means that for “most” functions in $W_{2,\mu}^r$, asymptotic optimal linear operators are (modulo a constant) as good as optimal nonlinear operators in the $L_{q,\mu}$ ($1 \leq q < \infty$) metric.

Remark 1.4. Comparing (1.7) with (2.15) in [27], we obtain that in the average case setting, the Fourier partial summation operators S_n are the asymptotically optimal linear operators in the $L_{q,\mu}$ metric if and only if $1 \leq q < 2 + 1/\mu$. This is completely different from the one-dimensional periodic case and the spherical case, where the Fourier partial summation operators are the asymptotically optimal linear operators in the L_q metric for all q , $1 \leq q \leq \infty$ (see [31], [30], and [26]).

The paper is organized as follows. Section 2 contains some lemmas concerning probabilistic linear widths of diagonal matrixes in \mathbb{R}^m with the standard Gaussian measure. In Section 3, we give discretization theorems of estimates of probabilistic linear widths. Finally, based on the results obtained in Sections 3 and 2, we prove Theorems 1.1 and 1.2 in Section 4.

2. PROBABILISTIC LINEAR WIDTHS OF DIAGONAL MATRIXES ON \mathbb{R}^m

Let ℓ_q^m ($1 \leq q \leq \infty$) denote the m -dimensional normed space of vectors $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, with the ℓ_q^m norm

$$\|x\|_{\ell_q^m} := \begin{cases} \left(\sum_{i=1}^m |x_i|^q \right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \max_{1 \leq i \leq m} |x_i|, & q = \infty. \end{cases}$$

As usual, we identify \mathbb{R}^m with the space ℓ_2^m , use the notation $\langle x, y \rangle$ to denote the Euclidean inner product of $x, y \in \mathbb{R}^m$, and write $\|\cdot\|_2$ instead of $\|\cdot\|_{\ell_2^m}$. Consider in \mathbb{R}^m the standard Gaussian measure γ_m , which is given by

$$\gamma_m(G) = (2\pi)^{-m/2} \int_G \exp(-\|x\|_2^2/2) dx,$$

where G is any Borel subset in \mathbb{R}^m .

Let $1 \leq q \leq \infty$, $1 \leq n < m$, and $\delta \in [0, 1)$. Then the probabilistic linear (n, δ) -widths of a linear mapping $T : \mathbb{R}^m \rightarrow \ell_q^m$ is defined by

$$\lambda_{n,\delta}(T : \mathbb{R}^m \rightarrow \ell_q^m, \gamma_m) = \inf_{G_\delta} \inf_{T_n} \sup_{x \in \mathbb{R}^m \setminus G_\delta} \|Tx - T_n x\|_{\ell_q^m},$$

where G_δ runs over all possible Borel subset of \mathbb{R}^m with measure $\gamma_m(G_\delta) \leq \delta$, and T_n all linear operators from \mathbb{R}^m to ℓ_q^m with rank at most n .

There are a few results devoted to studying the probabilistic widths of a linear mapping $T : \mathbb{R}^m \rightarrow \ell_q^m$ (see [5, 7, 10, 16, 17, 19]). Throughout the section, we use the letter D to denote an $m \times m$ real diagonal matrix $\text{diag}(d_1, \dots, d_m)$ with $d_1 \geq d_2 \geq \dots \geq d_m > 0$, the letter D_n the diagonal matrix $\text{diag}(d_1, \dots, d_n, 0, \dots, 0)$ for $1 \leq n \leq m$, and the letter I_m the m by m identity matrix. Moreover, $\{e_1, \dots, e_m\}$ denotes the standard orthonormal basis in \mathbb{R}^m :

$$e_1 = (1, 0, \dots, 0), \dots, e_m = (0, \dots, 0, 1).$$

The following lemmas will be used in the proof of Theorem 1.1.

Lemma 2.1. (1) ([10]). *If $1 \leq q \leq 2$, $m \geq 2n$, $\delta \in (0, 1/2]$, then*

$$(2.1) \quad \lambda_{n,\delta}(I_m : \mathbb{R}^m \rightarrow \ell_q^m, \gamma_m) \asymp m^{1/q} + m^{1/q-1/2} \sqrt{\ln(1/\delta)}.$$

(2) ([17]). If $2 \leq q < \infty$, $m \geq 2n$, $\delta \in (0, 1/2]$, then

$$(2.2) \quad \lambda_{n,\delta}(I_m : \mathbb{R}^m \rightarrow \ell_q^m, \gamma_m) \asymp m^{1/q} + \sqrt{\ln(1/\delta)}.$$

(3) ([19]). If $q = \infty$, $m \geq 2n$, $\delta \in (0, 1/2]$, then

$$(2.3) \quad \lambda_{n,\delta}(I_m : \mathbb{R}^m \rightarrow \ell_q^m, \gamma_m) \asymp \sqrt{\ln((m-n)/\delta)} \asymp \sqrt{\ln m + \ln(1/\delta)}.$$

Lemma 2.2. Assume that

$$\sum_{i=1}^m d_i^\beta \leq C(m, \beta) \quad \text{for some } \beta > 0.$$

Then for $2 \leq q \leq \infty$, $m \geq 2n$, $\delta \in (0, 1/2]$, we have

$$(2.4) \quad \lambda_{n,\delta}(D : \mathbb{R}^m \rightarrow \ell_q^m, \gamma_m) \ll \left(\frac{C(m, \beta)}{n+1} \right)^{\frac{1}{\beta}} \cdot \begin{cases} m^{1/q} + \sqrt{\ln(1/\delta)}, & 2 \leq q < \infty, \\ \sqrt{\ln m + \ln(1/\delta)}, & q = \infty. \end{cases}$$

Proof. First we show that

$$(2.5) \quad \int_{\mathbb{R}^m} \|Dx - D_n x\|_{\ell_q^m} \gamma_m(dx) \ll \left(\frac{C(m, \beta)}{n+1} \right)^{\frac{1}{\beta}} \cdot \begin{cases} m^{1/q}, & 2 \leq q < \infty, \\ \sqrt{\ln m}, & q = \infty. \end{cases}$$

Indeed, for $2 \leq q < \infty$, it follows from [7, (2.9)] that

$$(2.6) \quad \left(\int_{\mathbb{R}^m} \|Dx - D_n x\|_{\ell_q^m}^q \gamma_m(dx) \right)^{1/q} = C(q) \left(\sum_{k=n+1}^m d_k^q \right)^{1/q},$$

where $C(q) = \left(\pi^{-\frac{1}{2}} 2^{\frac{q}{2}} \Gamma(\frac{q+1}{2}) \right)^{1/q}$. Since

$$(n+1)d_{n+1}^\beta \leq \sum_{i=1}^{n+1} d_i^\beta \leq \sum_{i=1}^m d_i^\beta \leq C(m, \beta),$$

we get

$$(2.7) \quad d_{n+1} \leq \left(\frac{C(m, \beta)}{n+1} \right)^{\frac{1}{\beta}}.$$

Hence, we have

$$\begin{aligned} \int_{\mathbb{R}^m} \|Dx - D_n x\|_{\ell_q^m} \gamma_m(dx) &\leq \left(\int_{\mathbb{R}^m} \|Dx - D_n x\|_{\ell_q^m}^q \gamma_m(dx) \right)^{1/q} \ll \left(\sum_{k=n+1}^m d_k^q \right)^{1/q} \\ &\leq d_{n+1}(m-n)^{1/q} \ll m^{1/q} \left(\frac{C(m, \beta)}{n+1} \right)^{\frac{1}{\beta}}, \end{aligned}$$

proving (2.5) for $2 \leq q < \infty$. For $q = \infty$, it follows from (2.6) and (2.7) that for any $1 < q_1 < \infty$,

$$\begin{aligned} \int_{\mathbb{R}^m} \|Dx - D_n x\|_{\ell_\infty^m} \gamma_m(dx) &\leq \left(\int_{\mathbb{R}^m} \|Dx - D_n x\|_{\ell_{q_1}^m}^{q_1} \gamma_m(dx) \right)^{1/q_1} \\ &= C(q_1) \left(\sum_{k=n+1}^m d_k^{q_1} \right)^{1/q_1} \leq C(q_1) d_{n+1} (m-n)^{1/q_1} \\ (2.8) \quad &\leq c m^{\frac{1}{q_1}} \left(\Gamma\left(\frac{q_1+1}{2}\right) \right)^{\frac{1}{q_1}} \left(\frac{C(m, \beta)}{n+1} \right)^{\frac{1}{\beta}}. \end{aligned}$$

By Stirling's formula (see [1, p. 18]):

$$\lim_{x \rightarrow +\infty} \frac{\Gamma(x)}{\sqrt{2\pi} x^{x-\frac{1}{2}} \exp(-x)} = 1,$$

we obtain

$$\left(\Gamma\left(\frac{x+1}{2}\right)\right)^{\frac{1}{x}} \leq c(\sqrt{2\pi})^{\frac{1}{x}} \left(\frac{x+1}{2}\right)^{\frac{1}{2}} \exp\left(-\frac{x+1}{2x}\right) \leq c x^{\frac{1}{2}}.$$

Hence, taking $q_1 = \ln(e^2 m)$, we obtain from (2.8) that

$$\int_{\mathbb{R}^m} \|Dx - D_n x\|_{\ell_\infty^m} \gamma_m(dx) \leq c m^{\frac{1}{q_1}} q_1^{\frac{1}{2}} \left(\frac{C(m, \beta)}{n+1}\right)^{\frac{1}{\beta}} \ll \sqrt{\ln m} \left(\frac{C(m, \beta)}{n+1}\right)^{\frac{1}{\beta}},$$

which completes the proof of (2.5).

Now we shall show (2.4). We need the following lemma.

Lemma 2.3. ([2, (1.7.7)], [21, p. 47]) *Let $F : \mathbb{R}^m \rightarrow \mathbb{R}$ be a function satisfying the following Lipschitz condition*

$$|F(x) - F(y)| \leq \sigma \|x - y\|_2, \quad x, y \in \mathbb{R}^m,$$

for some $\sigma > 0$ independent of x and y . If $X \sim N_m(0, I_m)_{\mathbb{R}^m}$ is an \mathbb{R}^m -valued Gaussian random vector with mean 0 and covariance matrix I_m , then for all $t > 0$

$$(2.9) \quad \mathbb{P}(|F(X) - \mathbb{E}F(X)| \geq t) \leq 2 \exp\left(-\frac{t^2}{K^2 \sigma^2}\right),$$

with $K > 0$ being an absolute constant. (2.9) is called the Gaussian concentration inequality.

We continue to prove Lemma 2.2. Let $F(x) = \|Dx - D_n x\|_{\ell_q^m}$ for $x \in \mathbb{R}^m$ and $2 \leq q \leq \infty$. By (2.7) we obtain for any $x, y \in \mathbb{R}^m$

$$\begin{aligned} |F(x) - F(y)| &\leq \|(D - D_n)(x - y)\|_{\ell_q^m} = \left(\sum_{k=n+1}^m d_k^q |x_k - y_k|^q\right)^{\frac{1}{q}} \\ &\leq d_{n+1} \left(\sum_{k=n+1}^m |x_k - y_k|^q\right)^{\frac{1}{q}} \leq d_{n+1} \left(\sum_{k=n+1}^m |x_k - y_k|^2\right)^{\frac{1}{2}} \\ &\leq \left(\frac{C(m, \beta)}{n+1}\right)^{\frac{1}{\beta}} \|x - y\|_2 =: \sigma_1 \|x - y\|_2, \end{aligned}$$

where $\sigma_1 := \left(\frac{C(m, \beta)}{n+1}\right)^{\frac{1}{\beta}}$. Thus, applying the Gaussian concentration inequality (2.9) yields

$$\mathbb{P}(|F(X) - \mathbb{E}F(X)| \geq t) \leq 2 \exp\left(-\frac{t^2}{K^2 \sigma_1^2}\right), \quad \forall t > 0,$$

where, here and in what follows, $X \sim N_m(0, I_m)_{\mathbb{R}^m}$. In particular, this implies that for $Q_\delta = \{x \in \mathbb{R}^m : F(x) > \mathbb{E}F(X) + K\sigma_1 \sqrt{\ln(2/\delta)}\}$ with $\delta \in (0, 1)$,

$$\gamma_m(Q_\delta) \leq \mathbb{P}(|F(X) - \mathbb{E}F(X)| > K\sigma_1 \sqrt{\ln(2/\delta)}) \leq \delta.$$

By the definition of the linear (n, δ) widths, this last equation further implies that

$$\begin{aligned} \lambda_{n, \delta}(D : \mathbb{R}^m \rightarrow \ell_q^m, \gamma_m) &\leq \sup_{\mathbb{R}^m \setminus Q_\delta} \|Dx - D_n x\|_q \\ (2.10) \quad &\leq \mathbb{E}F(X) + K\sigma_1 \sqrt{\ln(2/\delta)}. \end{aligned}$$

On the other hand, however, using (2.5), we have

$$(2.11) \quad \mathbb{E}F(X) = \mathbb{E}\|DX - D_n X\|_{\ell_q^m} = \int_{\mathbb{R}^m} \|Dx - D_n x\|_{\ell_q^m} \gamma_m(dx) =: \sigma_2,$$

where σ_2 denotes the right expression of (2.5). Thus, combining (2.10) with (2.11), we deduce the desired upper estimates. \square

3. DISCRETIZATION OF THE PROBABILISTIC LINEAR WIDTHS

This section is devoted to obtaining the discretization theorems which give the reduction of the calculation of probabilistic widths of a given function class to the computation of probabilistic widths of a finite-dimensional set equipped with the standard Gaussian measure.

Let η be a nonnegative C^∞ -function on $[0, +\infty)$ supported in $[0, 2]$ and equal to 1 on $[0, 1]$. We also suppose that $\eta(x) > 0$ for $x \in [0, 2]$. We define

$$(3.1) \quad L_{n,\eta}(x, y) := \sum_{j=0}^{\infty} \eta\left(\frac{j}{n}\right) P_j(x, y), \quad x, y \in \mathbb{B}^d,$$

where $P_k(x, y)$ is given in (1.3). It is well known that for any $P \in \Pi_n^d$,

$$(3.2) \quad \int_{\mathbb{B}^d} L_{n,\eta}(x, y) P(y) W_\mu(y) dy = P(x), \quad x \in \mathbb{B}^d.$$

We recall that the operator S_n is the orthogonal projection operator from $L_{2,\mu}$ to Π_n^d , i.e.,

$$S_n f(x) = \sum_{k=0}^n Proj_k(f)(x) = \sum_{k=0}^n \int_{\mathbb{B}^d} f(y) P_k(x, y) W_\mu(y) dy,$$

where $Proj_k$ is the orthogonal projector from $L_{2,\mu}$ onto \mathcal{V}_k^d defined by (1.2). For any $f \in L_{2,\mu}$, we define

$$(3.3) \quad \delta_1(f) = S_2(f), \quad \delta_k(f) = S_{2^k}(f) - S_{2^{k-1}}(f) \quad \text{for } k = 2, 3, \dots$$

Then for $x \in \mathbb{B}^d$,

$$\delta_k(f)(x) = \sum_{j=2^{k-1}+1}^{2^k} Proj_j(f)(x) = \langle f, M_k(\cdot, x) \rangle,$$

where

$$(3.4) \quad M_k(x, y) = \sum_{j=2^{k-1}+1}^{2^k} \sum_{i=1}^{a_j^d} \phi_{ji}(x) \phi_{ji}(y) = \sum_{j=2^{k-1}+1}^{2^k} P_k(x, y)$$

is the reproducing kernel of the Hilbert space $L_{2,\mu} \cap \left(\bigoplus_{j=2^{k-1}+1}^{2^k} \mathcal{V}_j^d \right)$, and $\{\phi_{j1}, \dots, \phi_{ja_j^d}\}$

is an orthonormal basis for \mathcal{V}_j^d . Obviously, for any $x \in \mathbb{B}^d$, $M_k(\cdot, x) \in \bigoplus_{j=2^{k-1}+1}^{2^k} \mathcal{V}_j^d$

and for any $f \in \bigoplus_{j=2^{k-1}+1}^{2^k} \mathcal{V}_j^d$, $f(x) = \delta_k(f)(x) = \langle f, M_k(\cdot, x) \rangle$. In special, for any $x, y \in \mathbb{B}^d$,

$$\langle M_k(\cdot, x), M_k(\cdot, y) \rangle = M_k(x, y).$$

Now we introduce the metric $\tilde{\rho}$ on \mathbb{B}^d :

$$\tilde{\rho}(x, y) := \arccos \left((x, y) + \sqrt{1 - \|x\|_2^2} \sqrt{1 - \|y\|_2^2} \right), \quad x, y \in \mathbb{B}^d.$$

For $r > 0$, $x \in \mathbb{B}^d$, we set $B_{\tilde{\rho}}(x, r) := \{y \in \mathbb{B}^d \mid \tilde{\rho}(x, y) \leq r\}$. For $\delta > 0$, $n \in \mathbb{N}$, we say that a finite subset $\Lambda \subset \mathbb{B}^d$ is maximal $(\frac{\delta}{n}, \tilde{\rho})$ -separated if

$$\mathbb{B}^d \subset \bigcup_{y \in \Lambda} B_{\tilde{\rho}}(y, \frac{\delta}{n}) \quad \text{and} \quad \min_{y \neq y' \in \Lambda} \tilde{\rho}(y, y') \geq \frac{\delta}{n}.$$

The following cubature formulae and Marcinkiewicz-Zygmund (MZ) inequalities on \mathbb{B}^d are crucial for establishing discretization theorems.

Lemma 3.1. (see [6, 20]) *There exists a constant $\gamma > 0$ depending only on d and μ such that for any $\delta \in (0, \gamma]$, any positive integer n , and any maximal $(\delta/n, \tilde{\rho})$ -separated subset $\Lambda \subset \mathbb{B}^d$, there exists a sequence of positive numbers*

$\omega_\xi \asymp n^{-d}(\frac{1}{n} + \sqrt{1 - \|\xi\|_2^2})^{2\mu}$, $\xi \in \Lambda$, for which the following quadrature formula holds for all $f \in \Pi_{4n}^d$,

$$\int_{\mathbb{B}^d} f(x) W_\mu(x) dx = \sum_{\xi \in \Lambda} \omega_\xi f(\xi).$$

Moreover, for any $1 \leq q \leq \infty$, $f \in \Pi_n^d$, we have

$$\|f\|_{q, \mu} \asymp \begin{cases} (\sum_{\xi \in \Lambda} |f(\xi)|^q \omega_\xi)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \max_{\xi \in \Lambda} |f(\xi)|, & q = \infty, \end{cases}$$

where the constants of equivalence depend only on d and μ .

Lemma 3.2. (see [27, Lemma 3]) *Let $\mu > 0$, $\beta \in (0, 1/(2\mu))$ and let Λ , ω_ξ be given as in Lemma 3.1. Then*

$$\sum_{\xi \in \Lambda} \omega_\xi^{-\beta} \ll n^{d(1+\beta)}.$$

For $k = 1, 2, \dots$, let $\gamma > 0$ be the same as in Lemma 3.1 and let $\Lambda_k = \{\xi_1, \dots, \xi_{u_k}\}$ be a maximal $(\gamma 2^{-(k+2)}, \tilde{\rho})$ -separated subset of \mathbb{B}^d . It is easy to know that $u_k \asymp 2^{kd}$. By Lemma 3.1, there exists a sequence of positive numbers $w_i \asymp 2^{-kd}(2^{-k} + \sqrt{1 - \|\xi_i\|_2^2})^{2\mu}$, $1 \leq i \leq u_k$, for which the following quadrature formula holds for all $f \in \Pi_{2^{k+4}}^d$,

$$(3.5) \quad \int_{\mathbb{B}^d} f(x) W_\mu(x) dx = \sum_{i=1}^{u_k} w_i f(\xi_i).$$

Moreover, for any $1 \leq q \leq \infty$, $f \in \Pi_{2^{k+2}}^d$, we have

$$\|f\|_q \asymp \left(\sum_{i=1}^{u_k} |f(\xi_i)|^q \omega_i \right)^{1/q} = \|U_k(f)\|_{\ell_{q, \omega}^{u_k}},$$

where $U_k : \Pi_{2^{k+2}}^d \mapsto \mathbb{R}^{u_k}$ is defined by

$$(3.6) \quad U_k(f) = (f(\xi_1), \dots, f(\xi_{u_k})),$$

and for $x \in \mathbb{R}^{u_k}$,

$$\|x\|_{\ell_{q,\omega}^{u_k}} := \begin{cases} \left(\sum_{i=1}^{u_k} |x_i|^q \omega_i \right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \max_{1 \leq i \leq u_k} |x_i|, & q = \infty. \end{cases}$$

Next, we define the operator $T_k : \mathbb{R}^{u_k} \mapsto \Pi_{2^{k+1}}^d$ by

$$(3.7) \quad T_k a(x) := \sum_{i=1}^{u_k} a_i w_i L_{2^k, \eta}(x, \xi_i),$$

where $a = (a_1, \dots, a_{u_k}) \in \mathbb{R}^{u_k}$, $L_{n, \eta}$ is defined as in (3.1). It is shown in [27, (2.15)] that for any q , $1 \leq q \leq \infty$,

$$(3.8) \quad \|T_k a\|_{q, \mu} \ll \|a\|_{\ell_{q, \omega}^{u_k}}.$$

For $f \in \Pi_{2^k}^d$, by (3.2) and (3.5) we get

$$f(x) = \int_{\mathbb{B}^d} f(y) L_{2^k, \eta}(x, y) d\sigma(y) = \sum_{j=1}^{u_k} w_j f(\xi_j) L_{2^k, \eta}(x, \xi_j),$$

which means that $f = T_k U_k f$ for any $f \in \Pi_{2^k}^d$. We also use the letters S_k , R_k , V_k to denote $u_k \times u_k$ real diagonal matrixes as follows:

$$(3.9) \quad S_k = \text{diag}(\omega_1^{\frac{1}{2}}, \dots, \omega_1^{\frac{1}{2}}), \quad R_k = \text{diag}(\omega_1^{\frac{1}{q}}, \dots, \omega_1^{\frac{1}{q}}), \quad V_k = \text{diag}(\omega_1^{-\frac{1}{2} + \frac{1}{q}}, \dots, \omega_1^{-\frac{1}{2} + \frac{1}{q}}),$$

and use the letter R_k^{-1} to represent the inverse matrix of R_k . Clearly, for $x \in \mathbb{R}^{u_k}$,

$$\|R_k x\|_{\ell_q^{u_k}} = \|x\|_{\ell_{q, \omega}^{u_k}} \quad \text{and} \quad R_k = V_k S_k.$$

Lemma 3.3. *For any $z = (z_1, \dots, z_{u_k}) \in \mathbb{R}^{u_k}$, we have*

$$(3.10) \quad \left\| \sum_{j=1}^{u_k} \omega_j^{1/2} z_j M_k(\cdot, \xi_j) \right\|_{2, \mu} \ll \|z\|_{\ell_2^{u_k}},$$

where $M_k(x, y)$ is given in (3.4), and $\{\xi_1, \dots, \xi_{u_k}\}$ is defined as above.

Proof. Denote by K the set $\{g \in \bigoplus_{j=2^{k-1}+1}^{2^k} \mathcal{V}_j^d \mid \|g\|_{2, \mu} \leq 1\}$. Since

$$\sum_{j=1}^{u_k} \omega_j^{1/2} z_j M_k(\cdot, \xi_j) \in L_{2, \mu} \cap \left(\bigoplus_{j=2^{k-1}+1}^{2^k} \mathcal{V}_j^d \right),$$

By the Riesz representation theorem and Cauchy-Schwarz inequality we have

$$\begin{aligned} \left\| \sum_{j=1}^{u_k} \omega_j^{1/2} z_j M_k(\cdot, \xi_j) \right\|_{2, \mu} &= \sup_{g \in K} \left| \left\langle \sum_{j=1}^{u_k} \omega_j^{1/2} z_j M_k(\cdot, \xi_j), g \right\rangle \right| = \sup_{g \in K} \left| \sum_{j=1}^{u_k} \omega_j^{1/2} z_j g(\xi_j) \right| \\ &\leq \sup_{g \in K} \left(\sum_{j=1}^{u_k} |z_j|^2 \right)^{1/2} \left(\sum_{j=1}^{u_k} |g(\xi_j)|^2 \omega_j \right)^{1/2} \\ &\ll \sup_{g \in K} \left(\sum_{j=1}^{u_k} |z_j|^2 \right)^{1/2} \|g\|_{2, \mu} \leq \|z\|_{\ell_2^{u_k}}, \end{aligned}$$

which proves (3.10). \square

Theorem 3.4. *Let $1 \leq q \leq \infty$, $\sigma \in (0, 1)$, and let the sequences of numbers $\{n_k\}$ and $\{\sigma_k\}$ be such that $0 \leq n_k \leq u_k \asymp 2^{kd}$, $\sum_{k=1}^{\infty} n_k \leq n$, $\sigma_k \in (0, 1)$, $\sum_{k=1}^{\infty} \sigma_k \leq \delta$. Then*

$$(3.11) \quad \lambda_{n,\sigma}(W_{2,\mu}^r, \nu, L_{q,\mu}) \ll \sum_{k=1}^{\infty} 2^{-k\rho} \lambda_{n_k,\sigma_k}(V_k : \mathbb{R}^{u_k} \rightarrow \ell_q^{u_k}, \gamma_{u_k}).$$

Proof. For convenience, we write

$$\lambda_{n_k,\sigma_k} := \lambda_{n_k,\sigma_k}(V_k : \mathbb{R}^{u_k} \rightarrow \ell_q^{u_k}, \gamma_{u_k}),$$

where γ_{u_k} is the standard Gaussian measure in \mathbb{R}^{u_k} . Denote by L_k a linear operator from \mathbb{R}^{u_k} to \mathbb{R}^{u_k} such that the rank of L_k is at most n_k and

$$\gamma_{u_k}(\{y \in \mathbb{R}^{u_k} \mid \|V_k y - L_k y\|_{\ell_q^{u_k}} > 2\lambda_{n_k,\sigma_k}\}) \leq \sigma_k.$$

Then for any $f \in W_{2,\mu}^r$, by (3.8) we have

$$(3.12) \quad \begin{aligned} \|\delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f)\|_{q,\mu} &= \|T_k U_k \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f)\|_{q,\mu} \\ &\ll \|U_k \delta_k(f) - R_k^{-1} L_k S_k U_k \delta_k(f)\|_{\ell_{q,\omega}^{u_k}} \\ &= \|V_k S_k U_k \delta_k(f) - L_k S_k U_k \delta_k(f)\|_{\ell_q^{u_k}}, \end{aligned}$$

where δ_k , U_k , T_k , and S_k , V_k , R_k are defined by (3.3), (3.6), (3.7), and (3.9), respectively. Denote $y = S_k U_k \delta_k(f) = (\omega_1^{1/2} \delta_k(f)(\xi_1), \dots, \omega_{u_k}^{1/2} \delta_k(f)(\xi_{u_k})) \in \mathbb{R}^{u_k}$. Note that for $x \in \mathbb{B}^d$,

$$\delta_k(f)(x) = \langle f, M_k(\cdot, x) \rangle = \langle f^{(-r)}, M_k^{(-r,0)}(\cdot, x) \rangle_r = \langle f, M_k^{(-2r,0)}(\cdot, x) \rangle_r,$$

where $M_k^{(r_1,0)}(x, y)$ is the r_1 -order partial derivative of $M_k(x, y)$ with respect to the variable x , $r_1 \in \mathbb{R}$. By the property of Gaussian measures we know that if a random vector f in $W_{2,\mu}^r$ is a centered Gaussian random vector with covariance operator C_ν , then the vector

$$y = S_k U_k \delta_k(f) = (\langle f, \omega_1^{1/2} M_k^{(-2r,0)}(\cdot, \xi_1) \rangle_r, \dots, \langle f, \omega_{u_k}^{1/2} M_k^{(-2r,0)}(\cdot, \xi_{u_k}) \rangle_r)$$

in \mathbb{R}^{u_k} is a random vector with a centered Gaussian distribution γ in \mathbb{R}^{u_k} and its covariance matrix C_γ is given by

$$C_\gamma = \left(\langle C_\nu(\omega_i^{1/2} M_k^{(-2r,0)}(\cdot, \xi_i)), \omega_j^{1/2} M_k^{(-2r,0)}(\cdot, \xi_j) \rangle_r \right)_{i,j=1}^{u_k}.$$

Note that

$$\begin{aligned} &\left\langle C_\nu(\omega_i^{1/2} M_k^{(-2r,0)}(\cdot, \xi_i)), \omega_j^{1/2} M_k^{(-2r,0)}(\cdot, \xi_j) \right\rangle_r \\ &= \left\langle \omega_i^{1/2} M_k^{(-2r-s,0)}(\cdot, \xi_i), \omega_j^{1/2} M_k^{(-2r,0)}(\cdot, \xi_j) \right\rangle_r \\ &= \left\langle \omega_i^{1/2} M_k^{(-\rho,0)}(\cdot, \xi_i), \omega_j^{1/2} M_k^{(-\rho,0)}(\cdot, \xi_j) \right\rangle. \end{aligned}$$

Since for any $z = (z_1, \dots, z_{u_k}) \in \mathbb{R}^{u_k}$,

$$\sum_{j=1}^{u_k} \omega_j^{1/2} z_j M_k(\cdot, \xi_j) \in \bigoplus_{j=2^{k-1}+1}^{2^k} \mathcal{V}_j^d,$$

by (3.10) we get

$$\begin{aligned}
\int_{\mathbb{R}^{u_k}} (y, z)^2 \gamma(dy) &= z C_\gamma z^T = \sum_{i,j=1}^{u_k} z_i z_j \langle \omega_i^{1/2} M_k^{(-\rho,0)}(\cdot, \xi_i), \omega_j^{1/2} M_k^{(-\rho,0)}(\cdot, \xi_j) \rangle \\
&= \left\langle \sum_{j=1}^{u_k} \omega_j^{1/2} z_j M_k^{(-\rho,0)}(\cdot, \xi_j), \sum_{j=1}^{u_k} \omega_j^{1/2} z_j M_k^{(-\rho,0)}(\cdot, \xi_j) \right\rangle \\
&= \left\| \sum_{j=1}^{u_k} \omega_j^{1/2} z_j M_k^{(-\rho,0)}(\cdot, \xi_j) \right\|_2^2 \asymp 2^{-2k\rho} \left\| \sum_{j=1}^{u_k} \omega_j^{1/2} z_j M_k(\cdot, \xi_j) \right\|_2^2 \\
(3.13) \quad &\ll 2^{-2k\rho} \|z\|_{\ell_2^{u_k}}^2 = 2^{-2k\rho} \int_{\mathbb{R}^{u_k}} (y, z)^2 \gamma_{u_k}(dy).
\end{aligned}$$

Now consider the subset of $W_{2,\mu}^r$

$$G_k := \{f \in W_{2,\mu}^r \mid \|\delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f)\|_q > 2c_1 c_2 2^{-k\rho} \lambda_{n_k, \sigma_k}\},$$

where c_1, c_2 are the positive constants given in (3.12) and (3.13). Then it follows from (3.12) that

$$\begin{aligned}
\nu(G_k) &\leq \nu\left(\left\{f \in W_{2,\mu}^r \mid \|V_k S_k U_k \delta_k(f) - L_k S_k U_k \delta_k(f)\|_{\ell_q^{u_k}} > 2c_2 2^{-k\rho} \lambda_{n_k, \sigma_k}\right\}\right) \\
&= \gamma\left(\left\{y \in \mathbb{R}^{u_k} \mid \|V_k y - L_k y\|_{\ell_q^{u_k}} > 2c_2 2^{-k\rho} \lambda_{n_k, \sigma_k}\right\}\right).
\end{aligned}$$

By Theorem 1.8.9 in [2, p. 29], we know that if γ_1 and γ_2 are two centered Gaussian measures on \mathbb{R}^N and satisfy

$$\int_{\mathbb{R}^N} (y, x)^2 \gamma_1(dx) \geq \int_{\mathbb{R}^N} (y, x)^2 \gamma_2(dx), \quad \forall y \in \mathbb{R}^N,$$

then for every convex symmetric set E , $\gamma_1(E) \leq \gamma_2(E)$. Note that for any $t > 0$, the set $\{y \in \mathbb{R}^{u_k} \mid \|V_k y - L_k y\|_{\ell_q^{u_k}} \leq t\}$ is convex symmetric. It follows from (3.13) that

$$\begin{aligned}
\nu(G_k) &\leq \gamma\left(\left\{y \in \mathbb{R}^{u_k} \mid \|V_k y - L_k y\|_{\ell_q^{u_k}} > 2c_2 2^{-k\rho} \lambda_{n_k, \sigma_k}\right\}\right) \\
&\leq \lambda\left(\left\{y \in \mathbb{R}^{u_k} \mid \|V_k y - L_k y\|_{\ell_q^{u_k}} > 2c_2 2^{-k\rho} \lambda_{n_k, \sigma_k}\right\}\right) \\
&= \gamma_{u_k}\left(\left\{y \in \mathbb{R}^{u_k} \mid \|V_k y - L_k y\|_{\ell_q^{u_k}} > 2\lambda_{n_k, \sigma_k}\right\}\right) \leq \sigma_k,
\end{aligned}$$

where λ is a centered Gaussian measure in \mathbb{R}^{u_k} with covariance matrix $c_2^2 2^{-2k\rho} I_{u_k}$, I_{u_k} is the identity matrix in \mathbb{R}^{u_k} . Let us consider the set $G = \bigcup_{k=1}^\infty G_k$ and the linear operator \widetilde{T}_n on $W_{2,\mu}^r$ which is given by

$$\widetilde{T}_n f = \sum_{k=1}^\infty T_k R_k^{-1} L_k S_k U_k \delta_k(f).$$

From the hypothesis of the theorem, we get that

$$\nu(G) \leq \sum_{k=1}^\infty \nu(G_k) \leq \sum_{k=1}^\infty \sigma_k \leq \sigma,$$

and

$$\text{rank } \widetilde{T}_n \leq \sum_{k=1}^\infty \text{rank}(T_k R_k^{-1} L_k S_k U_k \delta_k) \leq \sum_{k=1}^\infty n_k \leq n.$$

Consequently, by the definitions of G , \widetilde{T}_n , $\{G_k\}$, and $\{L_k\}$,

$$\begin{aligned} \lambda_{n,\delta} \left(W_{2,\mu}^T, \nu, L_{q,\mu} \right) &\leq \sup_{f \in W_{2,\mu}^T \setminus G} \|f - \widetilde{T}_n f\|_{q,\mu} \\ &\leq \sup_{f \in W_{2,\mu}^T \setminus G} \sum_{k=1}^{\infty} \left\| \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \right\|_{q,\mu} \\ &\leq \sum_{k=1}^{\infty} \sup_{f \in W_{2,\mu}^T \setminus G_k} \left\| \delta_k(f) - T_k R_k^{-1} L_k S_k U_k \delta_k(f) \right\|_{q,\mu} \\ &\ll \sum_{k=1}^{\infty} 2^{-k\rho} \lambda_{n_k, \delta_k}, \end{aligned}$$

which completes the proof of Theorem 3.4. \square

Next we consider the lower estimates. We assume that $m \geq 6$ and $b_1 m^d \leq n \leq 2b_1 m^d$ with $b_1 > 0$ being independent of n and m . We let $\{x_j\}_{j=1}^N \subset \{x \in \mathbb{B}^d \mid \|x\|_2 \leq 2/3\}$ such that $N \asymp m^d$ and

$$\{x \in \mathbb{B}^d \mid \|x - x_j\|_2 \leq 2/m\} \cap \{x \in \mathbb{B}^d \mid \|x - x_i\|_2 \leq 2/m\} = \emptyset, \quad \text{if } i \neq j.$$

Obviously, such points x_j exist. We may take $b_1 > 0$ sufficiently large so that $N \geq 2n$. Let φ^1 be a C^∞ -function on \mathbb{R}^d supported in $\{x \in \mathbb{R}^d \mid \|x\|_2 \leq 1\}$ and be equal to 1 on $\{x \in \mathbb{R}^d \mid \|x\|_2 \leq 2/3\}$, and let φ^2 be a nonnegative C^∞ -function on \mathbb{R}^d supported in $\{x \in \mathbb{R}^d \mid \|x\|_2 \leq 1/2\}$ and be equal to 1 on $\{x \in \mathbb{R}^d \mid \|x\|_2 \leq 1/4\}$. We define

$$\varphi_i(x) = \varphi^1(m(x - x_i)) - c_i \varphi^2(m(x - x_i)),$$

for some c_i such that $\int_{\mathbb{B}^d} \varphi_i(x) W_\mu(x) dx = 0$, $i = 1, \dots, N$. We set

$$A_N := \text{span} \{\varphi_1, \dots, \varphi_N\} = \left\{ F_{\mathbf{a}}(x) = \sum_{j=1}^N a_j \varphi_j(x) : \mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N \right\}.$$

Clearly,

$$\varphi_j \in W_{2,\mu}^T, \quad \text{supp } \varphi_j \subset \{x \in \mathbb{B}^d \mid \|x - x_j\|_2 \leq 1/m\} \subset \{x \in \mathbb{B}^d \mid \|x\|_2 \leq 5/6\},$$

$$\|\varphi_j\|_{q,\mu} \asymp \left(\int_{\mathbb{B}^d} |\varphi_j(x)|^q dx \right)^{1/q} \asymp m^{-d/q}, \quad 1 \leq q \leq \infty, \quad j = 1, \dots, N,$$

and

$$\text{supp } \varphi_j \cap \text{supp } \varphi_i = \emptyset \quad (i \neq j).$$

Hence, if $F_{\mathbf{a}} \in A_N$, $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$, then

$$(3.14) \quad \|F_{\mathbf{a}}\|_{q,\mu} \asymp \left(N^{-1} \sum_{j=1}^N |a_j|^q \right)^{1/q} = m^{-d/q} \|\mathbf{a}\|_{\ell_q^N}.$$

For a positive integer $v = 0, 1, \dots$ and $F_{\mathbf{a}} \in A_N$, $\mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$, it follows from the definition of $-D_\mu^d$ that

$$\text{supp}(-D_\mu^d)^v(\varphi_j) \subset \{x \in \mathbb{B}^d \mid \|x - x_j\|_2 \leq 1/m\},$$

and

$$\|(-D_\mu^d)^v(\varphi_j)\|_{q,\mu} \ll m^{2v-d/q}.$$

Hence, for $1 \leq q \leq \infty$ and $F_{\mathbf{a}} = \sum_{j=1}^N a_j \varphi_j \in A_N$,

$$\|(-D_\mu^d)^v(F_{\mathbf{a}})\|_{q,\mu} \ll m^{2v-d/q} \|\mathbf{a}\|_{\ell_q^N}.$$

It then follows by the Kolmogorov type inequality (see [8, Theorem 8.1]) that

$$\begin{aligned} \|F_{\mathbf{a}}^{(\rho)}\|_{q,\mu} &= \|(-D_\mu^d)^{\rho/2}(F_{\mathbf{a}})\|_{q,\mu} \\ &\ll \|(-D_\mu^d)^{1+[\rho]}(F_{\mathbf{a}})\|_{q,\mu}^{\frac{\rho}{2+2[\rho]}} \|F_{\mathbf{a}}\|_{q,\mu}^{1-\frac{\rho}{2+2[\rho]}} \\ (3.15) \quad &\ll m^{\rho-d/q} \|\mathbf{a}\|_{\ell_q^N} \ll m^\rho \|F_{\mathbf{a}}\|_{q,\mu}. \end{aligned}$$

For $f \in L_{1,\mu}$ and $x \in \mathbb{B}^d$, we define

$$P_N(f)(x) = \sum_{j=1}^N \frac{\varphi_j(x)}{\|\varphi_j\|_{2,\mu}^2} \int_{\mathbb{B}^d} f(y) \varphi_j(y) W_\mu(y) dy$$

and

$$Q_N(f)(x) = \sum_{j=1}^N \frac{\varphi_j(x)}{\|\varphi_j\|_{2,\mu}^2} \int_{\mathbb{B}^d} f(y) \varphi_j^{(\rho)}(y) W_\mu(y) dy.$$

Obviously, the operator P_N is the orthogonal projector from $L_{2,\mu}$ to A_N , and if $f \in W_{2,\mu}^\rho$, then $Q_N(f)(x) = P_N(f^{(\rho)})(x)$. Also, it follows from [27] that P_N is the bounded operator from $L_{q,\mu}$ to $A_N \cap L_{q,\mu}$, i.e., for $1 \leq q \leq \infty$,

$$(3.16) \quad \|P_N(f)\|_{q,\mu} \ll \|f\|_{q,\mu}.$$

Since $Q_N(f) \in A_N$ for $f \in W_{2,\mu}^\rho$, by (3.15) we have

$$(3.17) \quad \|(Q_N(f))^{(\rho)}\|_{2,\mu} \ll m^\rho \|Q_N(f)\|_{2,\mu} = m^\rho \|P_N(f^{(\rho)})\|_{2,\mu} \ll m^\rho \|f^{(\rho)}\|_{2,\mu}.$$

Theorem 3.5. *Let $1 \leq q \leq \infty$, $\delta \in (0, 1)$, and let N be given above. Then*

$$\lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu}) \gg n^{-\rho/d+1/2-1/q} \lambda_{n,\delta}(I_N : \mathbb{R}^N \rightarrow \ell_q^N, \gamma_N),$$

where $N \asymp n$, $N \geq 2n$, I_N is the N by N identity matrix, and γ_N is the standard Gaussian measure in \mathbb{R}^N .

Proof. Let T_n be a bounded linear operator on $W_{2,\mu}^r$ with rank $T_n \leq n$ such that

$$\nu(\{f \in W_{2,\mu}^r \mid \|f - T_n f\|_{q,\mu} > 2\lambda_{n,\delta}\}) \leq \delta,$$

where $\lambda_{n,\delta} := \lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu})$. Note that if A is a bounded linear operator from $W_{2,\mu}^r$ to $W_{2,\mu}^r$ and from $H(\nu)$ to $H(\nu)$, then the image measure λ of ν under A is also a centered Gaussian measure on $W_{2,\mu}^r$ with covariance

$$R_\lambda(f)(f) = \langle A^* C_\nu f, A^* C_\nu f \rangle_{H(\nu)}, \quad f \in W_{2,\mu}^r,$$

where C_ν is the covariance of the measure ν , $H(\nu) = W_{2,\mu}^\rho$ is the Cameron-Martin space of ν , and A^* is the adjoint of A in $H(\nu)$ (see [2, Theorem 3.5.1, p. 112]). Furthermore, if the operator A also satisfies

$$\|Af\|_{H(\nu)} \leq \|f\|_{H(\nu)},$$

then

$$R_\lambda(f)(f) = \|A^* C_\nu f\|_{H(\nu)}^2 \leq \|A^*\|^2 \|C_\nu f\|_{H(\nu)}^2 \leq \langle C_\nu f, C_\nu f \rangle_{H(\nu)} = R_\nu(f)(f).$$

By Theorem 3.3.6 in [2, p. 107], we get that for any absolutely convex Borel set E of $W_{2,\mu}^r$, there holds inequality

$$\nu(E) \leq \lambda(E).$$

It follows from (3.17) that

$$\|Q_N(f)\|_{H(\nu)} = \|(Q_N(f))^{(\rho)}\|_{2,\mu} \ll m^\rho \|f^{(\rho)}\|_{2,\mu} = m^\rho \|f\|_{H(\nu)}.$$

Then there exists a positive constant c_3 such that

$$\left\| \frac{1}{c_3 m^\rho} Q_N(f) \right\|_{H(\nu)} \leq \|f\|_{H(\nu)}.$$

Note that for any $t > 0$, the set $\{f \in W_{2,\mu}^r \mid \|f - T_n f\|_{q,\mu} \leq t\}$ is absolutely convex. It then follows that

$$\begin{aligned} & \nu(\{f \in W_{2,\mu}^r \mid \|f - T_n f\|_{q,\mu} > 2\lambda_{n,\delta}\}) \\ & \geq \nu(\{f \in W_{2,\mu}^r \mid \|Q_N f - T_n Q_N f\|_{q,\mu} > 2c_3 m^\rho \lambda_{n,\delta}\}). \end{aligned}$$

Now we define the linear operators $L_N : \mathbb{R}^N \mapsto A_N$ and $J_N : A_N \mapsto \mathbb{R}^N$ by

$$L_N(\mathbf{a})(x) = \sum_{i=1}^N \frac{a_i \varphi_i(x)}{\|\varphi_i\|_{2,\mu}}, \quad \mathbf{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$$

and

$$J_N(F_{\mathbf{a}}) = (a_1 \|\varphi_1\|_{2,\mu}, \dots, a_N \|\varphi_N\|_{2,\mu}), \quad F_{\mathbf{a}} \in A_N,$$

respectively. Obviously, $L_N J_N(F_{\mathbf{a}}) = F_{\mathbf{a}}$ for any $F_{\mathbf{a}} \in A_N$. Set $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, where $y_j = \frac{1}{\|\varphi_j\|_{2,\mu}} \langle f, \varphi_j^{(\rho)} \rangle$. Then $y = J_N Q_N(f)$. It follows from (3.14) and the fact $\|\varphi_j\|_{2,\mu} \asymp m^{-d/2}$ that

$$(3.18) \quad \|L_N(\mathbf{a})\|_{q,\mu} \asymp \left(N^{-1} \sum_{j=1}^N \frac{|a_j|^q}{\|\varphi_j\|_{2,\mu}^q} \right)^{1/q} \asymp m^{-d/q+d/2} \|\mathbf{a}\|_{\ell_q^N}.$$

By (3.16) and (3.18), we know that for any $f \in W_{2,\mu}^r$,

$$\begin{aligned} \|Q_N(f) - T_n Q_N(f)\|_{q,\mu} & \gg \|P_N(Q_N(f)) - P_N T_n Q_N(f)\|_{q,\mu} \\ & = \|L_N J_N Q_N(f) - L_N J_N P_N T_n L_N J_N Q_N(f)\|_{q,\mu} \\ & \gg m^{-d/q+d/2} \|J_N Q_N(f) - J_N P_N T_n L_N J_N Q_N(f)\|_{\ell_q^N} \\ & \gg m^{-d/q+d/2} \|y - J_N P_N T_n L_N y\|_{\ell_q^N}. \end{aligned}$$

We remark that $g_k = \frac{\varphi_k}{\|\varphi_k\|_{2,\mu}}$, $k = 1, 2, \dots, N$ is an orthonormal system in $L_{2,\mu}$ and $g_k \in H(\nu) = W_{2,\mu}^\rho$. Then the random vector $(\langle f, g_1^{(\rho)} \rangle, \dots, \langle f, g_N^{(\rho)} \rangle) = y$ in \mathbb{R}^N on the measurable space $(W_{2,\mu}^r, \nu)$ has the standard Gaussian distribution γ_N in \mathbb{R}^N . It then follows that

$$\begin{aligned} & \nu(\{f \in W_{2,\mu}^r \mid \|Q_N f - T_n Q_N f\|_{q,\mu} > 2c_3 m^\rho \lambda_{n,\delta}\}) \\ & \geq \nu(\{f \in W_{2,\mu}^r \mid \|y - J_N P_N T_n L_N y\|_{\ell_q^N} > c_4 m^{\rho+d/q-d/2} \lambda_{n,\delta}\}) \\ & = \gamma_N(\{y \in \mathbb{R}^N \mid \|y - J_N P_N T_n L_N y\|_{\ell_q^N} > c_4 m^{\rho+d/q-d/2} \lambda_{n,\delta}\}) =: \gamma_N(G), \end{aligned}$$

where c_4 is a positive constant. Clearly, $\text{rank}(J_N P_N T_n L_N) \leq n$ and

$$\gamma_N(G) \leq \nu(\{f \in W_{2,\mu}^r \mid \|f - T_n f\|_{q,\mu} > 2\lambda_{n,\delta}\}) \leq \delta.$$

Therefore,

$$\lambda_{n,\delta}(I_N : \mathbb{R}^N \rightarrow \ell_q^N, \gamma_N) \leq \sup_{y \in \mathbb{R}^N \setminus G} \|y - J_N P_N T_n L_N y\|_{\ell_q^N} \ll m^{\rho+d/q-d/2} \lambda_{n,\delta}.$$

That is,

$$\lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu}) \gg n^{-\rho/d+1/2-1/q} \lambda_{n,\delta}(I_N : \mathbb{R}^N \rightarrow \ell_q^N, \gamma_N),$$

which completes the proof of Theorem 3.5. \square

4. PROOFS OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. The lower estimates for $\lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu})$ follow from Theorem 3.5, and (2.1), (2.2), and (2.3) for $1 \leq q \leq \infty$ immediately.

For the upper estimates for $\lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu})$ for $2 \leq q \leq \infty$, we use Theorem 3.4. For any fixed natural number n , assume $C_1 2^{md} \leq n \leq C_1^2 2^{md}$ with $C_1 > 0$ to be specified later. We may take sufficiently small positive numbers $\varepsilon > 0$ such that $\rho > \frac{d}{2} + (1+\varepsilon)(2+\varepsilon)\mu d(\frac{1}{2} - \frac{1}{q})$ and define

$$n_j = \begin{cases} u_j, & \text{if } j \leq m, \\ \left\lceil u_j 2^{d(1+\varepsilon)(m-j)-1} \right\rceil, & \text{if } j > m, \end{cases} \quad \text{and } \delta_j = \begin{cases} 0, & \text{if } j \leq m, \\ \delta 2^{m-j}, & \text{if } j > m, \end{cases}$$

where u_j is given in Theorem 3.4. Then

$$\sum_{j \geq 0} n_j \ll \sum_{j \leq m} 2^{jd} + \sum_{j > m} 2^{md(1+\varepsilon)-d\varepsilon j} \ll 2^{md},$$

and

$$\sum_{k=1}^{\infty} \delta_k \leq \delta.$$

Hence, we can take C_1 sufficiently large so that $\sum_{j=0}^{\infty} n_j \leq C_1 2^{md} \leq n$. It follows from Lemma 3.2 that for $\beta \in (0, \frac{1}{2\mu(1/2-1/q)})$, $2 \leq q \leq \infty$,

$$\sum_{j=1}^{u_k} \omega_j^{-\beta(1/2-1/q)} \ll 2^{kd} 2^{kd\beta(\frac{1}{2}-\frac{1}{q})}.$$

If $j \leq m$, then $n_j = u_j$, and thence $\lambda_{n_j, \delta_j}(V_j : \mathbb{R}^{u_j} \rightarrow \ell_q^{u_j}, \gamma_{u_j}) = 0$. If $j > m$, then taking $\frac{1}{\beta} = (2+\varepsilon)\mu(\frac{1}{2} - \frac{1}{q})$ and applying Lemma 2.2, we obtain for $2 \leq q < \infty$,

(4.1)

$$\lambda_{n_j, \delta_j}(V_j : \mathbb{R}^{u_j} \rightarrow \ell_q^{u_j}, \gamma_{u_j}) \ll 2^{jd(\frac{1}{2}-\frac{1}{q})-d(1+\varepsilon)(m-j)(2+\varepsilon)\mu(\frac{1}{2}-\frac{1}{q})} (2^{\frac{jd}{q}} + (\ln(\frac{1}{\delta}))^{\frac{1}{2}}),$$

and for $q = \infty$,

$$(4.2) \quad \lambda_{n_j, \delta_j}(V_j : \mathbb{R}^{u_j} \rightarrow \ell_q^{u_j}, \gamma_{u_j}) \ll 2^{jd/2-d(1+\varepsilon)(m-j)(2+\varepsilon)\mu/2} \sqrt{j + \ln(1/\delta)}.$$

Now we estimate the upper bounds for $\lambda_{n,\delta}(W_2^r, \mu, L_q)$ for $1 \leq q \leq \infty$. For $2 \leq q < \infty$, by (3.11) and (4.1) we get

$$\begin{aligned} & \lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu}) \\ & \ll \sum_{j=m+1}^{\infty} 2^{-j(\rho-\frac{d}{2}+\frac{d}{q})} 2^{-d(1+\varepsilon)(m-j)(2+\varepsilon)\mu(\frac{1}{2}-\frac{1}{q})} (2^{\frac{jd}{q}} + (\ln(\frac{1}{\delta}))^{\frac{1}{2}}) \\ & \ll 2^{-m(\rho-\frac{d}{2}+\frac{d}{q})} (2^{md/q} + (\ln(1/\delta))^{1/2}) \\ & \ll n^{-\rho/d+1/2} (1 + n^{-2/q} \ln(1/\delta))^{1/2}. \end{aligned}$$

For $q = \infty$, it follows from (3.11) and (4.2) that

$$\begin{aligned} \lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{\infty,\mu}) & \ll \sum_{j=m+1}^{\infty} 2^{-j\rho} 2^{jd/2-d(1+\varepsilon)(m-j)(2+\varepsilon)\mu/2} \sqrt{j + \ln(1/\delta)} \\ & \ll 2^{-m(\rho-d/2)} \sqrt{m + \ln(1/\delta)} \\ & \ll n^{-\rho/d+1/2} \sqrt{\ln(n/\delta)}. \end{aligned}$$

For $1 \leq q < 2$, we have

$$\lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu}) \leq \lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{2,\mu}) \ll n^{-\rho/d+1/2} (1 + n^{-1} \ln(1/\delta))^{1/2}.$$

The proof of Theorem 1.1 is complete. \square

Proof of Theorem 1.2. By the definition of $\lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu})$, there exists a linear operator L_n with rank $\leq n$ such that for any $\delta \in (0, 1/2]$ and some subset $G_\delta \subset W_{2,\mu}^r$ with $\nu(G_\delta) \leq \delta$,

$$\sup_{f \in W_{2,\mu}^r \setminus G_\delta} \|f - L_n f\|_{q,\mu} \leq 2\lambda_{n,\delta}(W_{2,\mu}^r, \nu, L_{q,\mu}).$$

Consider the sequence $\{G_{2^{-k}}\}_{k=0}^{\infty}$ of sets, where $G_1 = W_{2,\mu}^r$. Then it follows from (1.6) that

$$\begin{aligned} \lambda_n^{(a)}(W_{2,\mu}^r, \nu, L_{q,\mu})_p & \leq \left(\int_{W_{2,\mu}^r} \|f - L_n f\|_{q,\mu}^p \nu(df) \right)^{1/p} \\ & = \left(\sum_{k=0}^{\infty} \int_{G_{2^{-k}} \setminus G_{2^{-k-1}}} \|f - L_n f\|_{q,\mu}^p \nu(df) \right)^{1/p} \\ & \leq \left(\sum_{k=0}^{\infty} (2\lambda_{n,2^{-k-1}}(W_{2,\mu}^r, \nu, L_{q,\mu}))^p \nu(G_{2^{-k}}) \right)^{1/p} \\ & \ll \left(\sum_{k=0}^{\infty} 2^{-k} (\lambda_{n,2^{-k-1}}(W_{2,\mu}^r, \nu, L_{q,\mu}))^p \right)^{1/p} \\ & \ll \begin{cases} n^{-\rho/d+1/2}, & 1 \leq q < \infty, \\ n^{-\rho/d+1/2} \sqrt{\ln(en)}, & q = \infty. \end{cases} \end{aligned}$$

For the proof of the lower estimates for $\lambda_n^{(a)}(W_{2,\mu}^r, \nu, L_{q,\mu})_p$, let L_n be a linear operator from $L_{q,\mu}$ to $L_{q,\mu}$ with rank at most n . We set

$$G' = \{f \in W_{2,\mu}^r \mid \|f - L_n f\|_{q,\mu} \geq \frac{1}{2} \lambda_{n,1/e}(W_{2,\mu}^r, \nu, L_{q,\mu})\}.$$

Then $\nu(G') \geq 1/e$. Otherwise, if $\nu(G') < 1/e$, then by the definition of probabilistic linear (n, δ) -width, we have

$$\lambda_{n,1/e}(W_{2,\mu}^r, \nu, L_{q,\mu}) \leq \sup_{f \in W_{2,\mu}^r \setminus G'} \|f - L_n f\|_{q,\mu} \leq \frac{1}{2} \lambda_{n,1/e}(W_{2,\mu}^r, \nu, L_{q,\mu}),$$

which leads to a contradiction. Hence $\nu(G') \geq 1/e$. It follows from (1.6) that for $0 < p < \infty$,

$$\begin{aligned} \left(\int_{W_{2,\mu}^r} \|f - L_n f\|_{q,\mu}^p \nu(df) \right)^{1/p} &\geq \left(\int_{G'} \|f - L_n f\|_{q,\mu}^p \nu(df) \right)^{1/p} \\ &\geq \frac{1}{2} \lambda_{n,1/e}(W_{2,\mu}^r, \nu, L_{q,\mu}) (\nu(G'))^{1/p} \\ &\gg \begin{cases} n^{-\rho/d+1/2}, & 1 \leq q < \infty, \\ n^{-\rho/d+1/2} \sqrt{\ln(en)}, & q = \infty, \end{cases} \end{aligned}$$

which gives the required lower estimates for $\lambda_n^{(a)}(W_{2,\mu}^r, \nu, L_{q,\mu})_p$. This completes the proof of Theorem 1.2. \square

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